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On hydraulic control in a stratified fluid

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The conditions for hydraulic control to occur in a continuously stratified fluid are discussed, using density as a vertical coordinate in place of height. A suitable definition of Froude number, which varies with depth, is given. Three conditions for control emerge. One is that the flow be everywhere well-behaved; another is that control occurs when the local long-wave speed vanishes. These are shown to be equivalent. The location of the control is determined indirectly by the Froude number, which occurs as the coefficient in an ordinary differential equation; the Froude number must be somewhere less than a critical value for control to occur. The third condition requires the coalescence of two different solutions for the same boundary conditions at the point of control. It is shown that this requirement is non-trivial: examples given include a simple control by topography, a virtual control, and a control by a constriction. A direct connection with layered theory is produced. Brief discussions of bidirectional flow (where the isopycnal surface of zero velocity must be flat) and weak shocks are given.

1. Introduction

The flow of a fluid in a channel whose sides or floor are subject to variations has been a constant object of study for many decades. The solutions for the flow of a single homogeneous layer which passes either over a bump or through a constriction are treated in standard texts (e.g. Prandtl 1952). Extension of the problems to stratified fluids has been found difficult, unless the stratification is represented as a small number of layers, typically two. This restriction to layered flow has come about because of the intrinsic and awkward nonlinearities present in representations of stratified flow, save for special cases such as Long's (1953, 1955) solution, similarity solutions (e.g. Wood 1968; Benjamin 1981), or small-amplitude theories. Layered models have proved exceptionally useful in explaining observations from many areas (e.g. the treatment of ocean throughflow at Gibraltar by Armi & Farmer 1988). When only two layers are permitted, a solution technique using the two local Froude numbers gives a convenient simplification (Benton 1954; Armi 1986). For more layers, a Froude-number representation is still possible, although the matrix formulation that ensues is not simple to manipulate analytically (cf. Baines 1988; Baines & Guest 1988 for calculations with 64 layers; and Lawrence 1990 for a discussion of the need for multiple definitions of Froude number for a layered fluid). We give an alternative matrix formulation in §5. For continuous stratification, furthermore, it is far from clear which of many definitions of Froude number are in some sense optimal, as Baines' wide-ranging 1987 review makes clear. We provide here a definition of the Froude number relevant for a continuously stratified fluid;

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this definition is a function of the vertical coordinate, since *one* number cannot describe the behaviour of a stratified fluid save under special conditions.

In this paper we use density as a surrogate form of vertical coordinate and construct the hydraulic equations in that system. Several other coordinate systems have been used for hydraulics problems; Long (1953), for example, employs both an upstream (undisturbed) vertical coordinate and a vertical displacement, and Benjamin (1981) employs the upstream density profile to advantage. However, isopycnal coordinates are convenient for many applications since mass or density conservation along a streamline is automatically built in. It will also turn out that density coordinates rather naturally imply the role of the Froude number in hydraulic control. Using this coordinate system we then derive the conditions for the flow to be critical, or controlled. These conditions involve a simple ordinary differential equation whose solution depends solely on the vertical distribution of the Froude number. Criticality requires that the Froude number be less than a certain value, for most of the boundary conditions considered.

Using density coordinates implies some restrictions on the flow. One is that the stratification must be everywhere stable. Since solutions with low or negative Richardson numbers would be expected to be unstable anyway, this restriction is not strong. There is no formal reason why layered flows cannot be treated as a special case, and we shall connect the solutions found for a continuous fluid back to the more familiar layered model at several places in what follows. We shall also require the long-wave, or hydrostatic, approximation to hold. Without this approximation, density coordinates give a useful simplification, but the problem remains fully two-dimensional; thus lee-wave phenomena cannot easily be studied in this manner.

The horizontal velocity is assumed to be of one sign throughout, so that critical layers are not permitted. This is not a restriction imposed by our choice of coordinates. Bidirectional flows in a continuously stratified fluid (e.g. lock exchange flows) are somewhat peculiar, and do not behave like their layered counterpart unless the number of layers becomes very large. In particular, the surface at which the velocity is zero becomes everywhere flat in a continuously stratified fluid. Appendix A discusses this problem. Hydraulic jumps and instabilities are not considered in the main text, although Appendix B derives the weak jump conditions (following Su 1976) and shows how these are connected to the conditions on Froude number derived in the text.

Section 2 derives equations for a set of hydraulics problems using density coordinates. Section 3 discusses hydraulic control in the system, and §4 derives a necessary condition on the Froude number for control to occur. The connection with layered models is discussed in §5; specific examples are provided in §6.

2. Formulation

We consider the flow shown schematically in figure 1. A stably stratified fluid possesses density ρ. The density of the lowest stratum of fluid is ρ_0, and of the highest stratum $\rho_1 \equiv \rho_0 - \Delta\rho$. The fluid has velocity u in the x-direction, and runs along a channel of local width $b(x)$. The velocity varies with both x and ρ. The bottom of the channel is located at $z = h(x)$, where z is the vertical coordinate. The geometry of the channel varies sufficiently gradually that we may make the approximation that flow quantities do not vary significantly across the channel. The treatment is given for a non-Boussinesq fluid; to convert to the Boussinesq case, merely set ρ to a constant where it appears as a factor.

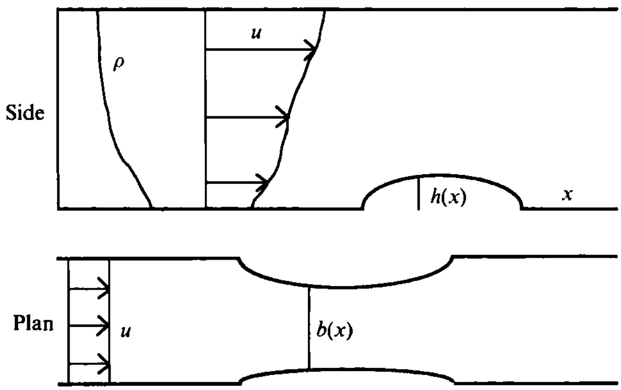

FIGURE 1. Side and plan views of the configuration in this paper.

We consider two upstream conditions:

Case (U1): a selective withdrawal problem from a reservoir of infinite width and specified stratification. Thus the width b becomes infinite far upstream, and the fluid velocity becomes zero there.

Case (U2): specified upstream flow and stratification in a channel whose width becomes uniform, of value b_0, and whose bottom perturbation vanishes.

We also consider three conditions at the fluid surface:

Case (S1): there is a rigid lid at height $z = H$.

Case (S2): there is an infinitely deep fluid of density ρ_1, i.e. the density of the highest stratum, occupying the space above the active fluid. The infinitely deep fluid is at rest.

Case (S3): there is an infinitely deep fluid density $\rho_1 - \delta\rho$ occupying the space above the active fluid. (This upper fluid could be air, corresponding to a free surface.) Thus a density jump of size $\delta\rho$ occurs at the top of the active fluid. This jump permits an external mode for the fluid, in which it behaves similar to a homogeneous fluid, and is largely controlled by the position of the surface (cf. Armi 1986 for a discussion of the two-layer case). We shall largely ignore this external mode in what follows.

In case (S3), when $\delta\rho$ is small compared with the density change within the active fluid $\Delta\rho$, we shall see that in a restricted sense (S3) becomes (S2), and when $\delta\rho$ becomes large with $\Delta\rho$, (S3) will become (S1).

As is traditional in use of density coordinates, we use the linear Bernoulli function B defined by

$$B = (\text{pressure} + \rho g z) \tag{1}$$

as a substitute for pressure, where g is the acceleration due to gravity. Then the hydrostatic relation yields

$$B_\rho = gz. \tag{2}$$

Conservation of mass of density ρ implies

$$\partial/\partial x\ (\rho u b z_\rho) = 0, \tag{3}$$

so that we may integrate with respect to x to give

$$\rho u b z_\rho = -Q(\rho), \tag{4}$$

where $Q(\rho)$ is the mass flux per unit density (and has units of volume flux). The sign is inserted to make the flux positive, since z_ρ is negative for stability. Combining (2) and (4) implies

$$ubB_{\rho\rho} = -\frac{gQ}{\rho}. \tag{5}$$

Whether the volume flux Q is a known function of density would usually depend on the upstream conditions. In case (U1), it would probably be unknown; in case (U2), it would be specified since the upstream velocity is known.

Conservation of energy takes a particularly simple form:

$$\partial/\partial x\,(\tfrac{1}{2}\rho u^2+B) = 0, \tag{6}$$

which also integrates immediately to

$$\tfrac{1}{2}\rho u^2+B = \tfrac{1}{2}\rho u_0^2+B_0, \tag{7}$$

where u_0, B_0 are both functions of ρ only. In case (U1), u_0 is identically zero since the flow speed vanishes in the reservoir. In both upstream cases, $B_0(\rho)$ is known from the boundary conditions in ρ given below, and the knowledge of the upstream stratification, since

$$B_{0\rho\rho} = gz_{0\rho} \tag{8}$$

and the upstream density gradient ρ_z satisfies

$$\rho_z = 1/z_\rho. \tag{9}$$

We also must define a Froude number for each value of density. A uniformly agreed definition of a Froude number is lacking, as Baines (1987) notes; cf. also Lawrence's (1990) discussion of a two-layer flow, which notes the relevance of four Froude numbers. For a layered model, there are (at least) as many Froude numbers as layers. Each takes the form

$$F^2 = u^2/(g'D),$$

where D is the thickness of the fluid layer and g' is a reduced gravity based on the density change across the layer. The values of the Froude numbers determine the properties of the flow, and in particular where criticality occurs for a particular fluid mode. (There are also as many modes as there are layers.) It is clear that only in special circumstances can a single number define flow properties for a continuous fluid (e.g. uniform flow and density gradient, such as Long's 1953 model).

Suppose we have n layers of fluid, and we let $n \to \infty$. Then $g' \to 0$ and $D \to 0$ also, making the value of the Froude number become infinite. Thus the appropriate definition for a continuously stratified fluid needs a suitable scaling: to avoid infinities; to provide the same value as used in other work when the fluid structure takes various special forms; and to give a value which bears directly on the criticality of the fluid flow. A logical choice for stratified fluids is to define a 'thickness' for a layer of infinitesimal thickness which is to be proportional to $-z_\rho$ (and indeed, z_ρ takes the role of thickness D in the equations of motion above). This no longer has units of depth, so must be rescaled. We choose, with foreknowledge of later sections, to scale with $\Delta\rho$, the density contrast across the fluid. (This effectively will define a Froude number for the internal modes of variability; a different scaling would be relevant for the external mode.) We also choose a reduced gravity g' as $g' = g\Delta\rho/\rho$. This then yields

$$F^2 = F^2(x,\rho) = -\frac{\rho u^2}{g\Delta\rho^2 z_\rho} = -\frac{\rho u^2}{\Delta\rho^2 B_{\rho\rho}}. \tag{10}$$

This definition reduces to Yih's (1965) value for uniform ρu^2 and density gradient, and also, apart from a π factor, to that used by Baines (1987). (The other traditional choice for F^2, a ratio between u and $(u-c)$, where c is a perturbation wave speed, cannot easily be defined because there are an infinite number of wave speeds to choose from, none of which is uniquely associated with any particular depth or density stratum.) The definition (10) is well-behaved, by the assumption of static stability; it would be hard to create a meaningful definition of a Froude number if the fluid were unstable.

Equations (5), (7) need boundary conditions. On the floor, (2) gives

$$B_\rho = gh, \quad \rho = \rho_0. \tag{11}$$

At the surface, we have

$$\left.\begin{aligned} &\text{case (S1)}: \quad B_\rho = gH, \quad &\rho = \rho_1; \\ &\text{case (S2)}: \quad B = 0, \quad &\rho = \rho_1; \\ &\text{case (S3)}: \quad B = \delta\rho B_\rho, \quad &\rho = \rho_1. \end{aligned}\right\} \tag{12}$$

The set (5), (7), (11) and (12) may usefully be combined into a single ordinary differential equation for the excess momentum flux

$$p = \rho(u^2 - u_0^2).$$

Then (5) gives

$$B_{\rho\rho} = -\frac{gQ}{b\rho(p\rho^{-1}+u_0^2)^{\frac{1}{2}}}, \tag{13}$$

so that differentiating (7) twice with respect to ρ and use of (13) implies

$$p_{\rho\rho} = 2\left\{\frac{gQ}{b\rho(p\rho^{-1}+u_0^2)^{\frac{1}{2}}} + B_{0\rho\rho}\right\}. \tag{14}$$

We shall frequently use u as a convenient shorthand for $(p\rho^{-1}+u_0^2)^{\frac{1}{2}}$ in the ensuing discussion. Then, in the upstream case (U2), u_0 is given by

$$u_0 = -\frac{gQ}{b_0\rho B_{0\rho\rho}} = \frac{QN_0^2}{b_0 g}, \tag{15}$$

where the buoyancy frequency N is given by

$$N^2 = -\frac{g\rho_z}{\rho}$$

and a subscript 0 again denotes values upstream.

Equation (14), plus boundary conditions derived below, would be used to solve for the flow; such solutions would typically be numerical. It is of interest, however, to re-express (14) using the Froude number. Substituting, we find that

$$p_{\rho\rho} - \frac{2p}{F^2\Delta\rho^2} = 2\left(B_{0\rho\rho} + \frac{\rho u_0^2}{\Delta\rho^2 F^2}\right) \tag{16}$$

in which the relevance of the scaling of F^2 becomes clear. Of course, F^2 involves p, so that (16) cannot be used for computations; the position is similar to the layered case in which F^2 again occurs, but the problem cannot be solved just from the terms involving F^2.

Equation (14) has two boundary conditions. At the floor, we have $B_\rho = gh$, $B_{0\rho} = 0$, so that differentiation of (7) gives

$$p_\rho = -2gh, \quad \rho = \rho_0. \tag{17}$$

At the surface, both B and B_0 satisfy one of the S conditions, so that p satisfies the homogeneous version of these:

$$\left.\begin{aligned} &\text{case (S1)}: \quad p_\rho = 0, \quad &\rho = \rho_1; \\ &\text{case (S2)}: \quad p = 0, \quad &\rho = \rho_1; \\ &\text{case (S3)}: \quad p = \delta\rho p_\rho, \quad &\rho = \rho_1. \end{aligned}\right\} \tag{18}$$

In (S3), (18) shows that under most conditions, $\delta\rho \to 0$ yields case (S2), and $\delta\rho \to \infty$ gives case (S1). However neither limit precludes the possibility of an external mode. The set (14), (17) and (18) may now in principle be solved at each value of x (and accompanying width b, height h) for p, from which the other quantities of interest may be derived. Note that the trivial solution for upstream conditions (U2), i.e. specified upstream flow, uniform width and a flat bottom, is

$$p \equiv 0,$$

corresponding to $u = u_0$.

The set (14), (17), and (18) may have no solution, a single solution, or multiple solutions. (We give examples below of 1, 2 and 3 solutions.) The solutions can be identified by a value of one of the variables, e.g. p or u at the upper boundary in case S1.

2.1. *Connection with other formulations*

It is straightforward to convert, e.g. Yih's (1965) expression of the Dubreil–Jacotin equation

$$\psi_{zz} + gz\frac{d\rho}{d\psi} = h(\psi),$$

to density coordinates. In Yih's formulation, ψ is the density-weighted streamfunction, so that $\psi_z = \rho^{\frac{1}{2}}u$. The transformation yields

$$\tfrac{1}{2}(\rho u^2)_\rho + gz = fn(\rho)$$

and the first differential of this gives (14). Thus Long's (1953) model, for example, is still a special case even in the density formulation, and can be derived simply.

3. Control

Hydraulic control is treated in the literature in at least three related ways. None seems to have been used to investigate fully stratified flow. The first method (e.g. Armi 1986) examines the x-derivative of the solution, and seeks conditions under which this is everywhere well-behaved. The second method identifies control points with locations where a small long-wave perturbation has zero phase velocity. The third (e.g. Gill 1977) regards the solution to (14) as part of a multivalued functional relationship between one parameter of the system and the geometrical parameters; at a control, the solution may be able to transfer smoothly from one branch of the relationship to another.

All approaches have advantages and disadvantages. We shall consider the first and second approach here, and show them to be equivalent, and postpone the third temporarily.

3.1. *Conditions for well-behaved solutions*

If we define

$$q = \partial p/\partial x \tag{19}$$

then differentiation with respect to x of (14) and its boundary conditions implies that q satisfies

$$q_{\rho\rho} + \frac{gQq}{b\rho^2u^3} = -\frac{2gQb_x}{\rho b^2 u}, \tag{20}$$

$$q_\rho = -2gh_x, \quad \rho = \rho_0, \tag{21}$$

and

$$\left.\begin{aligned} &\text{case (S1)}: \quad q_\rho(\rho_1) = 0; \\ &\text{case (S2)}: \quad q(\rho_1) = 0; \\ &\text{case (S3)}: \quad q(\rho_1) = \delta\rho q_\rho(\rho_1). \end{aligned}\right\} \tag{22}$$

From the definition of the Froude number, (20) can be rewritten in the form

$$q_{\rho\rho} + \frac{q}{F^2\Delta\rho^2} = -\frac{2gQb_x}{\rho b^2 u} = -\frac{2b_x}{b}\frac{(p+\rho u_0^2)}{F^2\Delta\rho^2}. \tag{23}$$

We will see below that at a controlled point, q satisfies the homogeneous version of (23). It is interesting to see how the role of F^2 has changed from the layered formulation. There, F entered a collection of algebraic equations for the flow variables. In the continuous formulation, F enters the problem as a spatially varying coefficient to an ordinary differential equation, and the way F varies will determine whether the flow is controlled, or even whether it has a solution.

At a non-controlled point in the fluid, the solution (22), (23) for q is well-behaved, and we enquire when this can break down. We solve (23) by variation of parameters. Write (23) as

$$\mathscr{L}(q) = -\frac{2b_x}{b}\frac{(p+\rho u_0^2)}{F^2\Delta\rho^2} = -\frac{2b_x}{b}\frac{\rho u^2}{F^2\Delta\rho^2} = \frac{2b_x}{b}B_{\rho\rho} \tag{24}$$

and let $r(\rho)$, $s(\rho)$ satisfy

$$\mathscr{L}(r) = \mathscr{L}(s) = 0, \tag{25}$$

together with initial conditions

$$\text{case (S1)}: \quad r(\rho_1) = 1, \quad r_\rho(\rho_1) = 0, \quad s(\rho_1) = 0, \quad s_\rho(\rho_1) = 1: \tag{26}$$

$$\text{case (S2)}: \quad r(\rho_1) = 0, \quad r_\rho(\rho_1) = 1, \quad s(\rho_1) = -1, \quad s_\rho(\rho_1) = 0; \tag{27}$$

$$\text{case (S3)}: \quad r(\rho_1) = 1, \quad r_\rho(\rho_1) = 1/\delta\rho, \quad s(\rho_1) = 0, \quad s_\rho(\rho_1) = 1. \tag{28}$$

Then thc Wronskian

$$rs_\rho - r_\rho s = 1 \tag{29}$$

in all cases, and the solution of (24) can be written

$$q = Ar + Cs. \tag{30}$$

Substitution yields as usual

$$A = A_0 + \frac{2b_x}{b\Delta\rho^2}\int_{\rho_1}^{\rho} \mathrm{d}\rho \frac{\rho u^2 s}{F^2}, \quad C = C_0 - \frac{2b_x}{b\Delta\rho^2}\int_{\rho_1}^{\rho} \mathrm{d}\rho \frac{\rho u^2 r}{F^2}, \tag{31}$$

where A_0, C_0 are constants. Substitution into (26), (27) or (28) requires that C_0 vanish in all cases, and condition (21) then gives

$$A_0 = \frac{1}{r_\rho(\rho_0)}\left\{-2gh_x + \frac{2b_x}{b\Delta\rho^2}s_\rho(\rho_0)\int_{\rho_1}^{\rho_0}\frac{\rho u^2 r}{F^2}\mathrm{d}\rho\right\} - \frac{2b_x}{b\Delta\rho^2}\int_{\rho_1}^{\rho_0}\frac{\rho u^2 s}{F^2}\mathrm{d}\rho. \tag{32}$$

This yields a well-behaved solution for q provided that $r_\rho(\rho_0)$ does not vanish. Equivalently, the solution is well-behaved provided that r does not satisfy the homogeneous equation for q, together with homogeneous boundary conditions.

When $r_\rho(\rho_0)$ vanishes, the flow is controlled. Since at such a point (29) implies that $s_\rho(\rho_0) = 1/r(\rho_0)$, there can only be a well-behaved solution for q if

$$h_x = \frac{b_x}{gr(\rho_0)\Delta\rho^2}\int_{\rho_1}^{\rho_0}\frac{\rho u^2 r}{F^2}\mathrm{d}\rho. \tag{33}$$

(Note that the integral can be written in many ways, e.g. as the integral of $rB_{\rho\rho}$.)

We have thus found two sets of conditions for criticality. One set relates to the flow geometry. For a flat bottom, h_x is identically zero. Hence a controlled, continuous solution can only occur when either

$$b_x = 0, \quad \text{i.e. a point of maximum constriction} \tag{34a}$$

or

$$\int_{\rho_1}^{\rho_0} \frac{\rho u^2 r}{F^2} \mathrm{d}\rho = 0. \tag{34b}$$

The latter will in general only occur at some discrete set of locations. Hence control can only occur at points of maximum constriction (34*a*), or at a set of virtual controls (34*b*). Armi (1986) and Wood (1968) show the same features for two-layer and similarity-solution continuous flows respectively. In the latter case (34*b*), for a given $Q(\rho)$, the fluid will not in general be able to reach some of the controls without a hydraulic jump occurring.

For a channel of uniform width, b_x is identically zero. In this case, a controlled solution can only occur when

$$h_x = 0, \quad \text{i.e. a point of maximum topographic height.} \tag{35}$$

This, too, is in agreement with Armi's (1986) results for two-layer flow. In particular, his comments concerning flows where the free surface is controlled apply: upstream of such a point, all internal modes must be at least critical, since there can be no controlled flow for any of the internal modes by (35). Furthermore, if one internal mode is controlled at a point of maximum height, then no other mode may be controlled upstream.

When both width and depth of the channel are permitted to vary, the requirement of control gives more complicated locations for the control points of the flow, and no simple deductions can be made.

The second criticality condition depends on the vertical distribution of the Froude number. Combining the features of the previous discussion, we can write this as

criticality implies that there exists a solution to

$$q_{\rho\rho} + \frac{q}{F^2 \Delta\rho^2} = 0$$

together with

$$q_\rho = 0, \ \rho = \rho_0, \text{ and condition (S1), (S2), or (S3).} \tag{36}$$

Note that (36) is a homogeneous equation for q. The condition it implies for F^2 is implicit, in that it involves the solution of an ordinary differential equation for q. We shall see below that the finite-difference version of this will reproduce the layered-model results.

Conditions (34)–(36) are necessary for control to occur, but not sufficient. Implicit in the idea of a flow passing from a subcritical to a supercritical regime, as noted earlier, is that there be two or more solutions for the same local geometry, and that two of these solutions merge at the point of control. This appears to be the case for the existing layer studies, and also applies for a continuously stratified fluid, as we shall show later. Also needed is that the shape of the geometry actually permits control to occur. (For example, in the uniform width one-layered case, u_x^2 is proportional to $-h_{xx}$ at the control. If h_{xx} is positive, corresponding to a depression, then control cannot occur even though the conditions on the first derivative have been met. Similarly, in the uniform depth one-layered case b_{xx} must be positive, corresponding to a constriction.)

Similar features occur in the continuous problem. If we define

$$v = q_x = p_{xx},$$

an equation for v may be derived by differentiating (23) with respect to x. This gives

$$v_{\rho\rho} + \frac{v}{F^2 \Delta\rho^2} = \sigma = \frac{1}{\Delta\rho^2 F^2}\left\{\frac{3q^2}{2\rho u^2} + \frac{2qb_x}{b} - \frac{2\rho u^2 b_{xx}}{b} + \frac{4\rho u^2 b_x^2}{b^2}\right\}. \tag{37}$$

Also, v satisfies boundary condition (S1), (S2), or (S3) at the surface and equals $-2gh_{xx}$ at the floor. The form of (37) is identical to (23), and solvability conditions can be found in the same way. In the general case these are not enlightening. In the two special cases mentioned above, however, (37) gives more information about the physics of the situation.

When the channel is everywhere flat ($h = 0$), and the flow passes through a point where b_x is zero, the shape of $q(\rho)$ has been determined, but not its amplitude, since the system is homogeneous. In this case σ reduces to

$$\frac{1}{\Delta\rho^2 F^2}\left\{\frac{3q^2}{2\rho u^2} - \frac{2\rho u^2 b_{xx}}{b}\right\},$$

and good behaviour requires that

$$\int_{\rho_1}^{\rho_0} r\sigma \,\mathrm{d}\rho = 0, \tag{38}$$

where r is the same (homogeneous) function as before. This requirement will normally define the amplitude of q, and hence yield the x-derivatives of the solution at the control, provided that q^2 is positive in (38). This in turn restricts the sign of b_{xx}; one would expect this to be positive on physical grounds, but it is not clear that this would always have to be so.

The other simple case is when the channel is of uniform width. As before, q satisfies a homogeneous equation and so its amplitude is not yet known. In this case, σ is simply

$$\frac{1}{\Delta\rho^2 F^2}\frac{3q^2}{2\rho u^2}.$$

Good behaviour now requires that

$$-2gh_{xx} - \frac{1}{r(\rho_0)}\int_{\rho_1}^{\rho_0} r\sigma \,\mathrm{d}\rho = 0. \tag{39}$$

Thus the amplitude of q is set by (39), again provided that the sign of h_{xx} is such that q^2 is positive.

The third (Gill) technique for hydraulics can be shown to be equivalent to the above arguments. Integrating (14) together with one of the S surface conditions (26)–(28) gives $p = p(\rho;\, b,\, d)$ where $\mathrm{d}(x)$ is an unknown function of integration. Application of (17) will then determine d unless $p_{\rho d}(\rho_0)$ vanishes. However, from (25), and (26)–(28), p_d is identically equal to r used above, whereupon the same conditions apply.†

3.2. *Long-wave perturbations*

We can also examine control by computing the long-wave speeds of small perturbations to flow moving locally with velocity $\bar{u}(\rho)$ and Bernoulli function $\bar{B}(\rho)$,

† I am indebted to a referee for this point.

in a fluid of uniform width and depth. Denoting time by t, the linearized equations for momentum and volume perturbations u, B become

$$\rho u_t + \rho \bar{u} u_x + B_x = 0, \tag{40}$$

$$B_{\rho\rho t} + \bar{u} B_{\rho\rho x} + \bar{B}_{\rho\rho} u_x = 0. \tag{41}$$

Seeking solutions which are functions of $(x-ct)$ implies that B satisfies

$$B_{\rho\rho} - \frac{\bar{B}_{\rho\rho} B}{\rho(\bar{u}-c)^2} = 0, \tag{42}$$

and boundary condition (12), plus

$$B_\rho(\rho_0) = 0. \tag{43}$$

(Recall that these are the conditions satisfied by r or q for control.) Using the definition of Q, (42) becomes

$$B_{\rho\rho} + \frac{gQ}{b\rho^2 \bar{u}(\bar{u}-c)^2} B = 0. \tag{44}$$

Now compare (44) with (36) noting that $p + \rho u_0^2 = \rho \bar{u}^2$, and the definition of Q. The two equations are identical when $c = 0$. In other words, the existence of a wave perturbation with zero phase speed is precisely equivalent to the condition of control deduced earlier, as expected. Now the system (44) possesses an infinity of wave speeds c. We also know, from Howard's (1961) argument, that the wave speeds are all real unless the Richardson number

$$Ri = \frac{N^2}{u_z^2} = -\frac{B_{\rho\rho}}{\rho u_\rho^2} < \tfrac{1}{4} \text{ somewhere.}\dagger$$

If the flow is everywhere stable, then the wave speeds in (44) are readily enumerable and (in theory, at least) we may also enumerate the control points of a flow. In practice, this is very difficult, and requires extensive numerical evaluation, especially for a channel of varying width. It seems unlikely that – save for special cases – one could produce a solution which was well-behaved all the way back to an infinitely wide channel, since the flow would have to satisfy an infinite number of critical conditions. These would of course determine the structure and amount of the mass flux Q. For flow in a channel of uniform width, where there is only one control point by (35), it is possible to construct solutions; one is given later.

4. A necessary condition for control

It is straightforward to derive a condition that the Froude number must satisfy for control to occur, at least for conditions (S1), (S2). At a control, q satisfies (36) and its homogeneous boundary conditions. We define a non-dimensional density coordinate

$$\eta = \frac{\rho - \rho_0}{\Delta\rho}, \tag{45}$$

which runs from -1 at the surface to 0 at the floor. Then (36) becomes

$$q_{\eta\eta} + \frac{q}{F^2} = 0, \tag{46}$$

together with

$$q_\eta = 0, \quad \eta = -1, 0. \tag{47}$$

† It is straightforward to use Howard's argument directly on (44) to produce the same result.

If we put $s = q_\eta$, we may rewrite this as

$$(F^2 s_\eta)_\eta + s = 0. \tag{48}$$

Multiplying by s and integrating from -1 to 0, we have in both cases (S1) and (S2)

$$\int_{-1}^{0} F^2 s_\eta^2 \, d\eta = \int_{-1}^{0} s^2 \, d\eta. \tag{49}$$

Now in case (S1), the second integral of (49) satisfies

$$\int_{-1}^{0} s^2 \, d\eta < \frac{1}{\pi^2} \int_{-1}^{0} s_\eta^2 \, d\eta, \tag{50}$$

since $s(-1)$ and $s(0)$ both vanish (cf. Hardy, Littlewood & Polya 1952, p. 185). Thus

$$\int_{-1}^{0} F^2 s_\eta^2 \, d\eta < \frac{1}{\pi^2} \int_{-1}^{0} s_\eta^2 \, d\eta, \tag{51}$$

so that

$$F < \frac{1}{\pi} \text{ somewhere for criticality to occur (S1).} \tag{52}$$

(This bound is achievable, by uniform flow with uniform density gradient. Baines (1987) gives a solution for this in z-coordinates, using a different definition of F. Yih (1965) discusses a uniform upstream flow for this value of F when the long-wave approximation is not made.) The form of (52) demonstrates the reasoning behind the scaling for F.

In case (S2), the condition on F^2 is slightly weaker. We now only have $s(-1) = 0$, and the relevant inequality is (Hardy *et al.* 1952, p. 184)

$$\int_{-1}^{0} F^2 s_\eta^2 \, d\eta = \int_{-1}^{0} s^2 \, d\eta < \frac{4}{\pi^2} \int_{-1}^{0} s_\eta^2 \, d\eta. \tag{53}$$

Accordingly we find that

$$F < \frac{2}{\pi} \text{ somewhere for criticality to occur (S2).} \tag{54}$$

There is no equivalent condition for condition (S3), since the boundary condition permits an external mode (i.e. one in which the fluid behaves more as a homogeneous, one-layer, fluid. (When $\delta\rho/\Delta\rho$ is large, there is a solution for which $F \sim \delta\rho/\Delta\rho \gg 1$, in fact. In this case the relevant Froude number would need to be scaled rather differently.)

These conditions may be combined into the single statement:

> *A necessary condition for control to occur at a given point in the channel, for conditions (S1) and (S2), is that F somewhere be less than some critical value F_c in the fluid column.*

If criticality with respect to higher internal modes is considered, then WKBJ theory can be applied to (36). It is clear that over at least part of the depth, F must be small, so that if the internal mode is sufficiently high, q can be written

$$q \approx \frac{1}{F^{\frac{1}{2}}} \cos \frac{1}{\Delta\rho} \int_{\rho_0}^{\rho} \frac{dq}{F}. \tag{55}$$

Application of the surface boundary conditions then yields

$$\text{case (S1)}: \quad \frac{1}{\Delta\rho}\int_{\rho_0}^{\rho}\frac{\mathrm{d}q}{F} \approx n\pi;$$

$$\text{case (S2)}: \quad \frac{1}{\Delta\rho}\int_{\rho_0}^{\rho}\frac{\mathrm{d}q}{F} \approx (n+\tfrac{1}{2})\,\pi. \tag{56}$$

As usual, these conditions even give a reasonable order-of-magnitude guide for low internal modes (the uniform width rigid-lid case considered in §6 has the integral in (56) equal to 4.54 at criticality, instead of π as required by (56)).

The necessary condition we have derived is far from being sufficient. It must be stressed that the entire vertical structure of the flow is involved with the determination of criticality, and in general (36) must be solved numerically.

5. Connection with layered models

It is of interest to connect the continuous representation of previous sections with layered-model representations. Consider for definiteness the rigid-lid surface condition (S1). If the density range (ρ_1, ρ_0) is divided into n subdivisions (which can be replaced by layers), each of density change $\Delta\rho/n$, then the solution q can be described by the vector $(q_1, q_2, \ldots, q_n)$, as in figure 2. Notice that the boundary conditions imply external values $q_0 = q_1$, $q_{n+1} = q_n$ at floor and surface respectively, and that the numbering is in the direction of decreasing density, to match standard usage (e.g. Baines 1988).

Then the second-order finite-difference representation of (36) becomes

$$\frac{q_{i+1}-2q_i+q_{i-1}}{(\Delta\rho/n)^2}+\frac{q_i}{F_i^2\Delta\rho^2} = 0, \quad i = 1, 2, \ldots, n. \tag{57}$$

Here F_i^2 is the value of F^2 at level i, given by

$$F_i^2 = -\frac{u_i^2\,\rho}{g z_\rho \Delta\rho^2},$$

and $\Delta\rho$ remains the density change across the entire fluid. Thus the density change across each layer, $\Delta\rho'$, is given by $\Delta\rho/n$, and the reduced gravity across each layer, g', is given by $g\Delta\rho'/\rho$. Finally, the depth of each layer, d_i, is equal to $-z_\rho\Delta\rho'$.

Thus the local (layered) Froude number f_i is given by

$$f_i^2 = \frac{u_i^2}{g'\,\mathrm{d}_i},$$

and the above reasoning shows that

$$F_i = \frac{f_i}{n}. \tag{58}$$

Then (57) becomes

$$q_{i+1}-2q_i+q_{i-1}+\frac{q_i}{f_i^2} = 0, \quad i = 1, 2, \ldots, n, \tag{59}$$

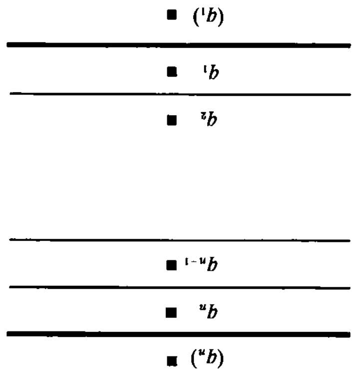

FIGURE 2. The finite-difference representation of the continuous problem.

which has non-zero solutions if

$$\begin{vmatrix} (1/f_1^2-1) & 1 & 0\ldots & & 0 \\ 1 & (1/f_2^2-2) & 1 & 0\ldots & 0 \\ 0 & 1 & (1/f_3^2-2) & 1\ldots & 0 \\ & & \ldots & & \\ 0 & 0 & \ldots & 1 & (1/f_n^2-1) \end{vmatrix} = 0. \tag{60}$$

If the surface conditions are altered, then the last two columns of (60) are modified accordingly.

In the case of two layers, (60) becomes

$$\begin{vmatrix} (1/f_1^2-1) & 1 \\ 1 & (1/f_2^2-1) \end{vmatrix} = 0, \tag{61}$$

i.e.

$$f_1^2+f_2^2 = 1, \tag{62}$$

which is the well-known requirement for rigid-lid control (Benton 1954; Armi 1986).

The entire density coordinate representation, indeed, converts immediately to the n-layer case by finite differencing (which is why, in some sense, density coordinates are natural for the problem). For example, consider the control problem with n fluid layers. We use Baines' (1988) formulation, in which the pressure in the top (nth) layer is p_s, and the depth and density of each layer are d_i, ρ_i respectively. Mass continuity implies that

$$\frac{\partial}{\partial x}(\rho_i u_i d_i) = 0, \tag{63}$$

so that, defining $\rho_i u_i^2$ as p_i to maintain notation,

$$(p_i d_i^2)_x = 0$$

or

$$d_{ix} = -\frac{d_i p_{ix}}{2p_i}. \tag{64}$$

We may also write the momentum equation at level i as

$$\tfrac{1}{2}p_{ix}+p_{sx}+g\sum_{j=i+1}^{n}\rho_j d_{jx}+g\sum_{j=0}^{i}\rho_i d_{ix} = 0. \tag{65}$$

We now take the second derivative of (65) in finite-difference form: take (65) evaluated at $(i+1)$ – twice (65) at (i) plus (65) at $(i-1)$, and use the fact that the ρ_i are linearly spaced with i. The result, after using (64), is

$$\tfrac{1}{2}(p_{i+1,x}-2p_{ix}+p_{i-1,x})+(\rho_{i+1}-\rho_i)g\left(\frac{-d_i}{2p_i}\right)p_{ix}=0. \tag{66}$$

Putting $q_i = p_{ix}$ as in the continuous case, we obtain

$$q_{i+1}-2q_i+q_{i-1}+\frac{g\Delta\rho' d_i}{\rho u_i^2}q_i=0, \tag{67}$$

which is precisely of the form (59). Exactly similar exercises can be performed for the top level, using the conditions (S1) or (S2). It is also straightforward to treat the layered equations themselves in this way, rather than in the more usual manner of Baines (1988), to obtain equations which are merely finite-difference equivalents of (16).

6. Some numerical examples

We have discussed control from two points of view: that control occurs at a point where the x-derivative of the flow is required to be well-behaved; and as the location of a point of vanishing long-wave perturbation velocity. These conditions are necessary for control, but not sufficient. To see this, we now consider, using numerical examples, the third requirement (e.g. Gill 1977) that there be two solutions for a given geometry, and that these link smoothly at the point of control.

All examples make the Boussinesq approximation, have a rigid lid (S1), and are non-dimensionalized (using a scale H for z, h; b_0 for b; $(g\Delta\rho H\rho_0^{-1})^{\frac{1}{2}}$ for u, and $(g\Delta\rho H)$ for B, so that the non-dimensional ρ runs from -1 at the surface to 0 at the floor like the η coordinate earlier).

6.1. *Hydraulic control over topography*

We choose the mass flux

$$Q = A+B\cos R(\rho+0.5), \tag{68}$$

which is symmetric about the midpoint of the density range. Here $R=4$, $A=0.3$, and $B=-0.21$. The upstream density gradient is uniform, so that $B_{0\rho\rho}=-1$. Thus the upstream velocity $u_0=Q$. The width b is maintained at unity.

When the topographic height h is less than $h_{\max}=0.10397$, there are two solutions to (14). One is subcritical, and one supercritical (i.e. all long-wave perturbation velocities are positive). When $h=h_{\max}$, the two solutions coincide; and there are no solutions for $h>h_{\max}$. Thus the control can only occur at $h=h_{\max}$, and from (35) this must be the maximum of the topographic height.

Figure 3 shows contours of density for the solution. Since the flow follows these contours, figure 3 also shows the contours of the streamfunction for the flow, although with unequal contour interval. On the supercritical side of the flow, there is a strong pycnocline at a height of about 0.15, and a rarefaction in the upper half of the fluid. Indeed, the Richardson number falls to 0.3 in that region, so that the flow is nearly unstable there. Figure 4 shows how the volume flux, initially symmetric about mid-depth, becomes increasingly concentrated at lower depths as the topography is passed, corresponding to the increased stratification at those depths. The u velocity (figure 5) is increased correspondingly. This yields very asymmetric profiles of F (figure 6); over the maximum of the topography, F is less than $1/\pi$ in

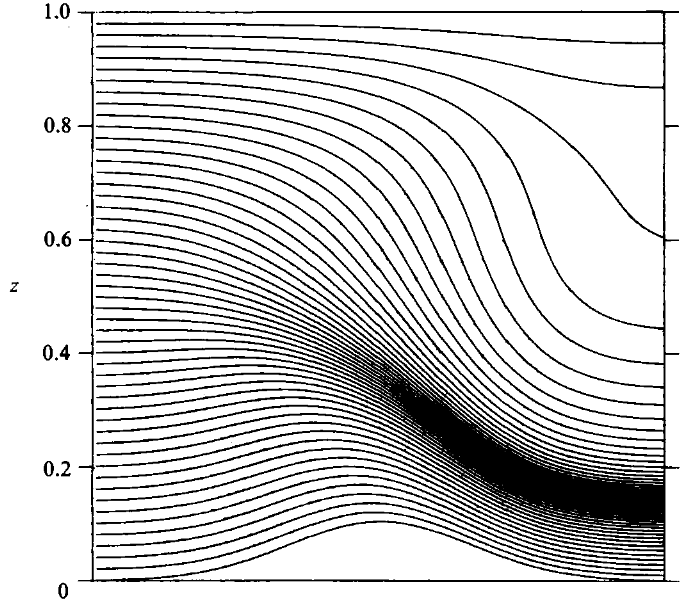

FIGURE 3. Contours of density, contour interval 0.02, for the problem (68), for flow in a channel of uniform width with a rigid lid. The lowest contour, $\rho = 0$, delineates the topography. The x-scale is arbitrary.

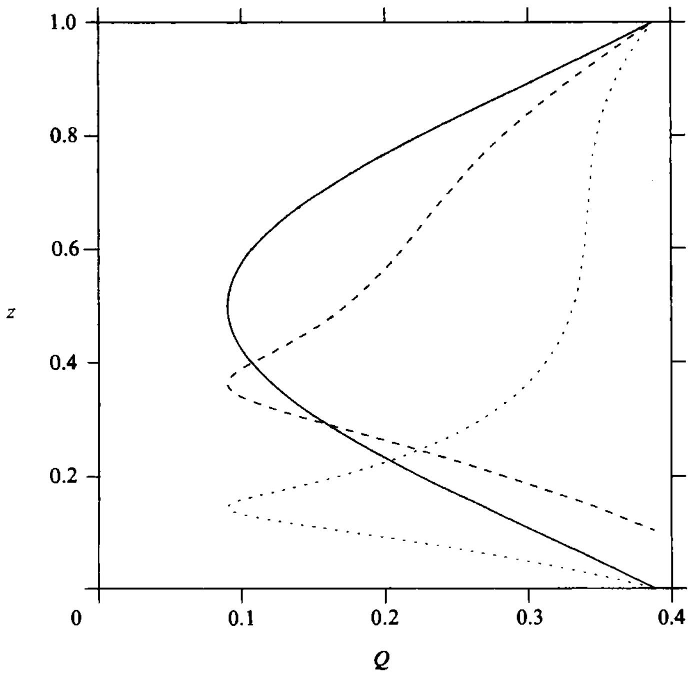

FIGURE 4. Plots of the volume flux Q for the problem in figure 3, shown here as a function of depth, —, upstream, ---, at the point of control, and ···, downstream. The curves all show the same function $Q(\rho)$.

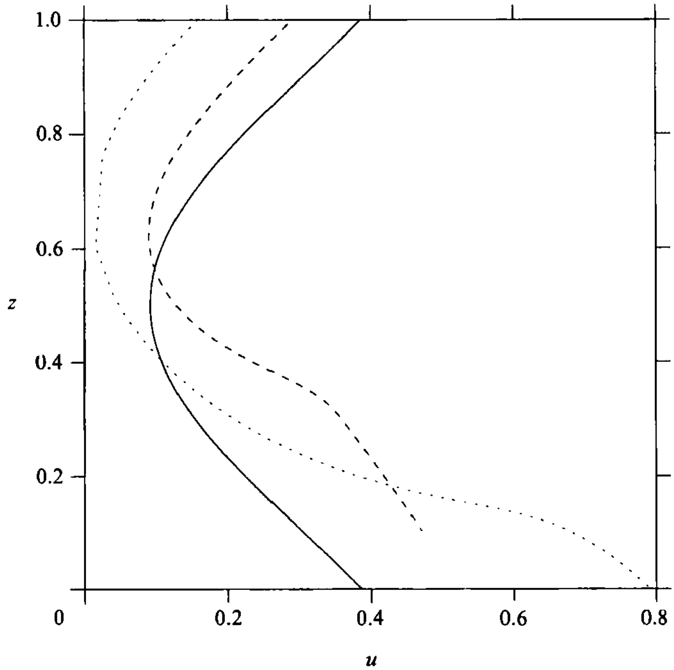

FIGURE 5. Plots of the velocity u for the problem in figure 3; annotation as for figure 4.

the depth range $0.52 \leqslant z \leqslant 0.79$, thus satisfying the necessary requirements for control.

6.2. *Virtual control through constrictions*

The same volume flux (68) and conditions are retained, but the topographic height h is now zero, and b is permitted to vary. As b decreases below unity, we find *three* solutions, shown in figure 7. Two (the left-hand and right-hand upper curves) are mirror images of each other in the vertical; recall that the vertical boundary conditions are symmetric for this problem. Both solutions are supercritical. The third solution, the middle curve in figure 7, is subcritical. The two supercritical solutions cease to exist for $b < b_c = 0.593$. The subcritical solution becomes critical at $b = b_c$, and supercritical for $b < b_c$, where there is only one solution. All solutions coincide at $b = b_c$.

The symmetry of the solution about $\rho = -0.5$ implies that F is also symmetric; thus conditions (34b) and (38) are both automatically satisfied at the point of control, just as in Wood's (1968) similarity solution. Hence $b = b_c$ is a point of virtual control. Figure 8 shows the solution for $1 \geqslant b \geqslant 0.2$.

It is also possible for the two supercritical branches to join smoothly, but only if $b = b_c$ is a local minimum in width, by (34a). This situation is shown in figure 9; the two solutions are mirror images of each other with respect to x and depth. Note that no other combinations are possible for control, since the smoothness of higher derivatives is only possible if the solutions themselves have continuous derivatives; equality of the two solutions is not sufficient. So, for example, neither of the supercritical branches can link smoothly to the subcritical solution at $b = b_c$ even at a local minimum of width.

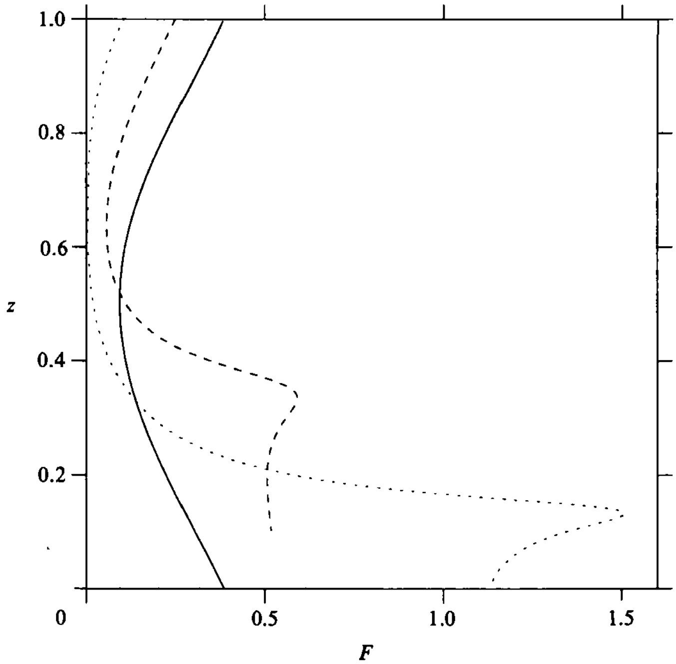

FIGURE 6. Plots of local Froude number F for the problem in figure 3; annotation as for figure 4.

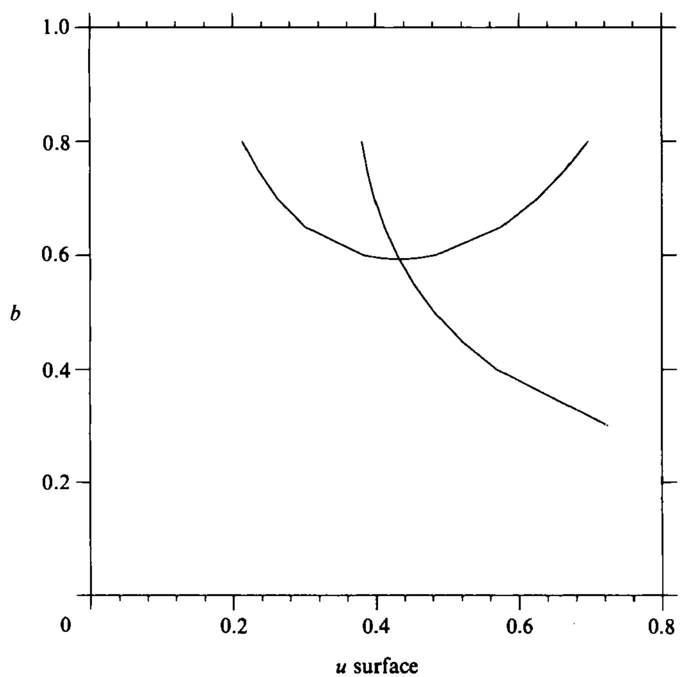

FIGURE 7. The surface velocity u_s for the mass flux Q in (68), as the channel width b varies in a channel of uniform depth. When b is larger than 0.593, there are three solutions. The central solution is subcritical, and continues smoothly through the virtual control. The other two solutions are supercritical, and are mirror images of each other in the vertical.

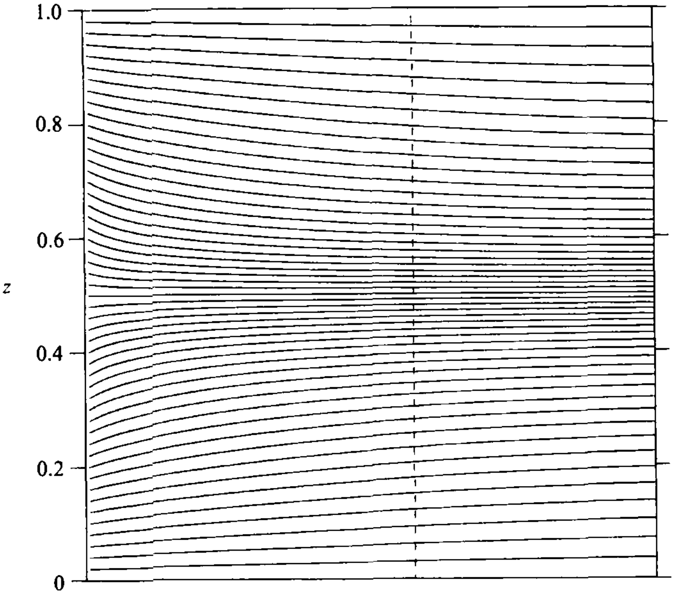

FIGURE 8. Contours of density for the problem in figure 7, for flow near the virtual control; b varies from 1 (at the left) to 0.2 (at the right). A vertical dotted line shows the virtual control. The x-scale is otherwise arbitrary.

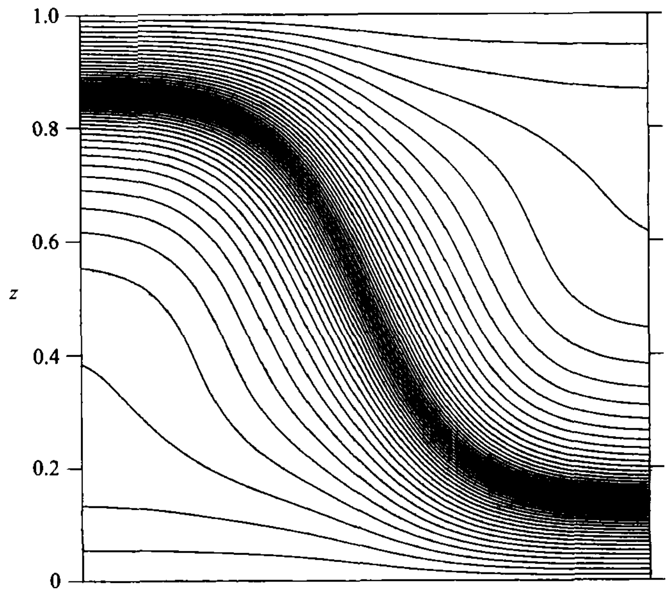

FIGURE 9. Contours of density for the problem in figure 7, for controlled flow between the two supercritical solutions (which are mirror images of each other). The width b ranges from 1 (at outer edges of the diagram to 0.593 at the centre, where the control occurs. The x-scale is otherwise arbitrary.

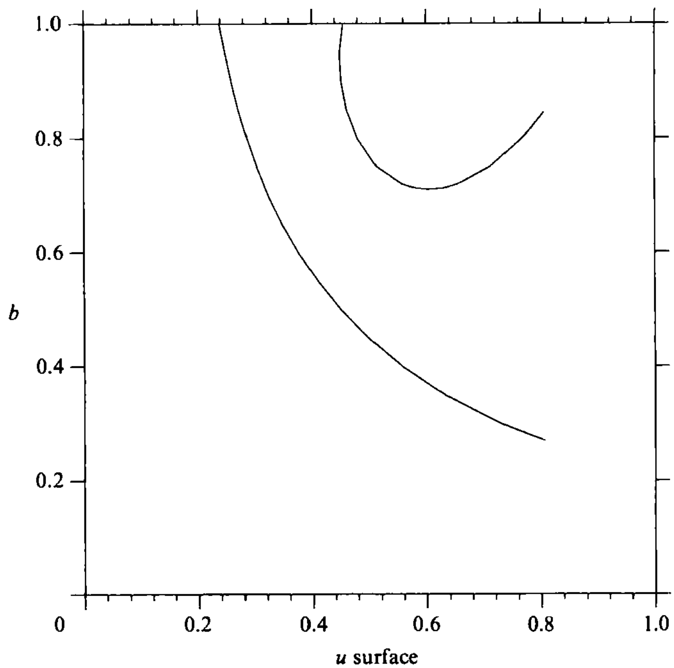

FIGURE 10. The surface velocity for mass flux Q given by (69), as width b varies. When $b > b_c = 0.71$, one subcritical and two supercritical solutions exist, although one of the supercritical solutions does not exist for $b > 0.84$. A normal control exists at $b = b_c$. Only one supercritical mode exists for $b < b_c$.

6.3. *Control through constrictions*

We modify the previous flux to make it asymmetric with respect to density. We use

$$Q = A + B\cos R(\rho + 0.4), \tag{69}$$

with A, B and R retaining their previous values. This removes the two symmetric supercritical modes of §6.2, and replaces them by a supercritical mode which is unaffected by the control, and another supercritical mode (which only exists for $b < 0.84$). The subcritical mode still exists; figure 10 shows the situation. When the width b reaches $b_c = 0.71$, the subcritical mode becomes critical and can join smoothly onto the second supercritical mode provided b is a minimum at b_c. When $b < b_c$, only the first supercritical mode remains.

8. Conclusions

The use of density coordinates has been shown to yield convenient simplifications in the study of stratified hydraulic theory, provided that the long-wave assumption holds. This paper has discussed some hydraulic control using these coordinates. A simple expression for hydraulic control has been derived, showing that, as in layered theory, contractions in channel width yield an infinite number of virtual controls, while topographic height changes only permit a single control, located at the maximum of the topography. A necessary condition for control, in terms of the local Froude number, has been derived. The theory connects straightforwardly with the more elaborate matrix theory when multiple layers are employed; in particular, the Benton–Armi expression for two-layer control can be derived by finite-differencing the continuous case.

A feature of the controlled solutions given here is that Richardson numbers often become small in supercritical regimes, implying that flow instabilities may be likely. This may be a generic feature. Wood's (1968) similarity solution is a case in point. In the density coordinate system, it is found very simply by seeking solutions of the form

$$p = \alpha(x)[B_0(\rho)+E\rho+F], \tag{70}$$

where the choice of E and F depend on the boundary conditions used.† This form, as noted by Wood (1968), automatically satisfies all conditions at each virtual control. The value α then satisfies an equation very similar to the problem for a single layer, as b varies, and can be written

$$\tfrac{1}{2}\alpha = 1-\frac{K}{b\alpha^{\frac{1}{2}}}, \tag{71}$$

where K is a constant dependant upon mass flux. Equation (71) has the familiar property of possessing two positive solutions for α or none, with control occurring for a particular value of b. If we evaluate the Richardson number Ri for these solutions, we find that Ri varies as a function of density times $(\alpha^{\frac{3}{2}}b)^{-1}$. Assuming that the problem involves flow from a very wide reservoir, through a contraction, and out into another wide reservoir, we see from (71) that $\alpha \sim 2$ for large b and supercritical flow, and $\alpha \sim (K/b)^2$ for large b and subcritical flow. Thus on the supercritical side, as b becomes small, Ri varies as b^{-1}, and becomes very small. This again suggests the presence of flow instabilities in the supercritical region.

David Andrews much improved the proof in §3; Jeff Blundell provided useful graphics; and the referees provided helpful comments.

Appendix A. Bidirectional flow

We examine here the properties of flow which takes both positive and negative values. The conclusions are derived using density coordinates, but may straightforwardly, if tediously, be found using the usual vertical coordinate.

We assume that $u(\rho)$ has a simple zero at $\rho = \rho_c$, and that z_ρ is well behaved there. Thus u passes smoothly from a positive to a negative value as the depth changes, and the density gradient remains finite. Then (3) implies that

$$u \equiv 0, \quad \rho = \rho_c, \text{ and hence } u_0 \equiv 0, \quad \rho = \rho_c, \tag{A 1}$$

so that the surface $\rho = \rho_c$ is motionless. Application of the Bernoulli constraint (7) then gives, using (A 1),

$$B(x, \rho_c) = B_0(\rho_c). \tag{A 2}$$

Differentiation of (7) with respect to ρ also gives

$$(\tfrac{1}{2}\rho u^2)_\rho + B_\rho = (\tfrac{1}{2}\rho u_0^2)_\rho + B_{0\rho}, \tag{A 3}$$

immediately implying

$$B_\rho(x, \rho_c) = B_{0\rho}(\rho_c), \tag{A 4}$$

or

$$z(x, \rho_c) = z_0(\rho_c). \tag{A 5}$$

Thus the surface of zero velocity is everywhere flat. (A 2) also shows that the pressure is uniform on the surface.

† The solution given by Wood relates to condition (S2), where the upper surface sinks as the fluid moves through a contraction. For case (S1), E and F are chosen to make the topographic height h be related to the solution structure: the topographic height rises through a constriction in case (S1) by precisely the amount that the upper surface sank under condition (S2).

A consequence of this is that both p ($\equiv \rho u^2$) and p_ρ vanish at $\rho = \rho_c$ so that the second-order equation (14) applies separately to two disconnected halves, separated by a rigid horizontal 'wall' on which horizontal and vertical velocity both vanish. This is not observed in nature (e.g. the Gibraltar data discussed by Armi & Farmer 1988) where it is usual for the surface dividing in- and outflow to migrate up and down, although most data refer to time-dependent situations.

The connection with layered fluids developed in §5 can also be used to show that as the number of layers becomes large (and the density jump between each layer becomes correspondingly small), then the layer with zero velocity – or its two neighbours if no layer has precisely zero velocity – becomes flat.

Other than because of time dependence, this peculiar response can only be removed by breaking one of the assumptions made. If u becomes discontinuous, then the resulting shear layer would almost certainly be unstable; the ensuing mixing would act to make u continuous once more. A jump in density does not give a problem in density coordinates (z_ρ becomes zero), and the argument goes through as before. Only if the density becomes homogenized at the point of zero velocity (so z_ρ becomes infinite) does the argument break down. This could be produced by mixing, perhaps induced by a critical layer at the point of flow reversal.

Appendix B. Weak shocks

It is enlightening to rederive the weak shock equations in this system, following Su (1976). We write, for a uniform width fluid, the time-varying system

$$\rho u_t + \rho u u_x + B_x = 0, \tag{B 1}$$

$$z_{\rho t} + (u z_\rho)_x = 0. \tag{B 2}$$

Combining these, we may write

$$\rho(u z_\rho)_t + \rho(u^2 z_\rho)_x + z_\rho B_x = 0. \tag{B 3}$$

Denote conditions up- and downstream of the shock by suffices u and d respectively. In a frame of reference moving with the shock, (B 2) implies

$$u_{\mathrm{u}} z_{\rho \mathrm{u}} = u_{\mathrm{d}} z_{\rho \mathrm{d}}, \tag{B 4}$$

and (B 3) becomes

$$\rho(u_{\mathrm{d}}^2 z_{\rho \mathrm{d}} - u_{\mathrm{u}}^2 z_{\rho \mathrm{u}}) + (B_{\mathrm{d}} - B_{\mathrm{u}}) \bar{z}_\rho = 0, \tag{B 5}$$

where the bar denotes a suitable average across the shock. Su (1976) takes this average to be one half of the up- and downstream values, and we do the same here. Using (B 4) to substitute for u_{d}, (B 5) gives

$$\rho\left(\frac{u_{\mathrm{u}}^2 z_{\rho \mathrm{u}}^2}{z_{\rho \mathrm{d}}} - u_{\mathrm{u}}^2 z_{\rho \mathrm{u}}\right) + (B_{\mathrm{d}} - B_{\mathrm{u}}) \tfrac{1}{2}(z_{\rho \mathrm{u}} + z_{\rho \mathrm{d}}) = 0. \tag{B 6}$$

We put

$$\xi(\rho) = \frac{z_{\rho \mathrm{d}}}{z_{\rho \mathrm{u}}} - 1, \tag{B 7}$$

which, substituted into (B 6), gives

$$-\frac{2\xi \rho u_{\mathrm{u}}^2}{(1+\xi)(2+\xi)} + (B_{\mathrm{d}} - B_{\mathrm{u}}) = 0. \tag{B 8}$$

Differentiating twice with respect to ρ then gives

$$-2\left[\frac{\rho u_{\mathrm{u}}^2 \xi}{(1+\xi)(2+\xi)}\right]_{\rho\rho} + g z_{\rho \mathrm{u}} \xi = 0, \tag{B 9}$$

which is almost identical to Su's (1976) result.

If the shock is very weak, $|\xi|$ is very small, and (B 9) becomes

$$-(\rho u_u^2 \xi)_{\rho\rho} + g z_{\rho u} \xi = 0. \qquad \text{(B 10)}$$

If we set $\rho u_u^2 \xi = q$, say, (B 10) becomes

$$q_{\rho\rho} + \frac{q}{F^2 \Delta\rho^2} = 0, \qquad \text{(B 11)}$$

which is (23) when there is no downstream variation. Now q also satisfies the same boundary conditions as for the control problem in §§3 and 4. This total agreement between a weak shock and $\partial/\partial x$ of the flow is to be expected, since the frame of reference is such that the shock is steady; hence control implies a wave perturbation of zero velocity.

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